Measuring statistical dependence and coupling of subsystems

Andreas Schmitz
Physics Department, University of Wuppertal,
D–42097 Wuppertal, Germany

We investigate recently proposed measures for the statistical dependence of systems with complex dynamical behavior. We consider appropriate model systems, to ensure that influences of individual properties of the systems are excluded. We demonstrate that it is indeed possible to obtain non–trivial directional information, but we also argue that the interpretation of this information is difficult.

The question for dependencies between measured signals occurs in many applications. First, it can be of interest for very practical reasons. This is e.g. the case if one wants to avoid the analysis of redundant information in a huge amount of measured data. On the other hand often physically different signals are taken from one system and the analysis of the dependence between them can give information of the hidden dynamics of the system. Finally dependency measures are useful tools when dealing with coupled systems and studying synchronization.

Beside the simple existence of dependencies further questions occur. These involve driver–response relationships, coupling directions and similar. To answer such questions, asymmetry of the used measure is a minimal requirement. Although many symmetric quantities can be de–symmetrized formally, e.g. by using delays, it is desirable to use measures that are asymmetric by construction. One class of such quantities was introduced recently [3,4] and we will discuss its properties in the later part of the paper.

The simplest and usually first method to search for dependencies is to consider linear correlations. Let X and Y be two random variables with expectation values $\bar{x} = E[X]$ and $\bar{y} = E[Y]$. The covariance $C(X,Y)$ is then given by the expectation value $E[(x-\bar{x})(y-\bar{y})]$. For time series the expectation values $E[\cdot]$ are estimated by averaging over time which yields the estimator

$$\hat{C}(X,Y) = \frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})\,, \tag{1}$$

for the covariance, with i being the time index. While from non–vanishing covariance a dependency between X and Y can be deduced, the opposite conclusion is not allowed. One possible generalization of Eq.(1) is to introduce a time lag τ between the two time series. With this, much additional information can be gained about dynamical properties of the underlying processes. To avoid the analysis of such vast information, we will only consider static, i.e. equal time, dependencies here. Time delayed generalizations of the measures used here are quite simple. Another useful modification of Eq.(1) is the introduction of powers different from one, making it nonlinear. The number of higher moments, however, proliferates so fast that the computational price for being able to find any statistical dependence is prohibitive.

Different approaches to analyze dependencies can be derived from information theory. A useful quantity is the mutual information

$$M(X,Y) = I(X) + I(Y) - I(X,Y)\,, \tag{2}$$

where $I(X), I(Y)$ are the respective Shannon entropies of the variables X and Y, and $I(X,Y)$ is the joint entropy. For independent systems the mutual information is zero. The entropies can be calculated with a partition scheme or with the generalized correlation sum [1].

Analyzing time series from two or more systems is often motivated by the presumption that the underlying systems are coupled in some way. Beside the strength of the coupling also the direction of the coupling is of interest. However, symmetric quantities like the mutual information cannot deliver information about the direction.

A new approach to extract mutual dependencies was presented recently [2,3]. There, the quantities are based on mean distances in phase resp. embedding space. Let $\mathbf{x}_i = (x_i, x_{i+\tau}, \ldots, x_{i+(m-1)\tau})$ be the N embedding vectors where m and τ are embedding dimension and the time delay. We will use $\tau = 1$ throughout the paper. Further let $r_i(j),\ j = 1,\ldots,k$, be the time indices of the k nearest neighbors of $\mathbf{x}_i$. The squared mean distance from these neighbors is then given by

$$R_i^{(k)}(X) = \frac{1}{k}\sum_{j=1}^{k}(\mathbf{x}_i - \mathbf{x}_{r_i(j)})^2\,. \tag{3}$$

Analogously, $R_i^{(k)}(Y)$ can be defined by exchanging X and Y. The corresponding time indices of the k nearest neighbors of $\mathbf{y}_i$ are called $s_i(j)$. Further, the *conditional* distance

$$R_i^{(k)}(X|Y) = \frac{1}{k}\sum_{j=1}^{k}(\mathbf{x}_i - \mathbf{x}_{s_i(j)})^2 \tag{4}$$

can be defined, which only differs from $R_i^{(k)}(X)$ in the indices used in the second term. For independent systems we expect $R_i^{(k)}(X|Y) \gg R_i^{(k)}(X)$, while strongly dependent systems yield $R_i^{(k)}(X|Y) \approx R_i^{(k)}(X)$. With these, a measure for dependence, like e.g.

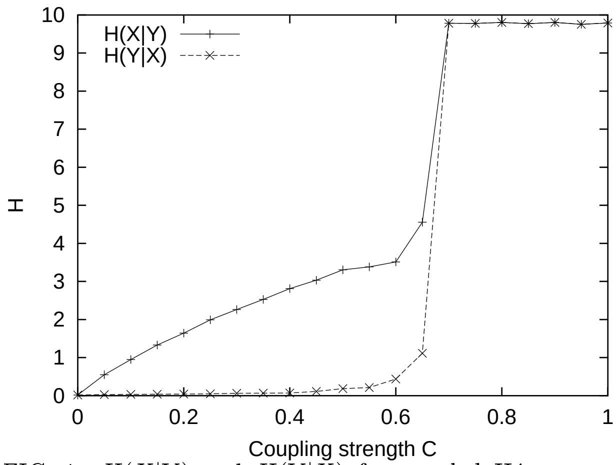

FIG. 1. $H(X|Y)$ and $H(Y|X)$ for coupled Hénon maps with $b_x = b_y = 0.3$.

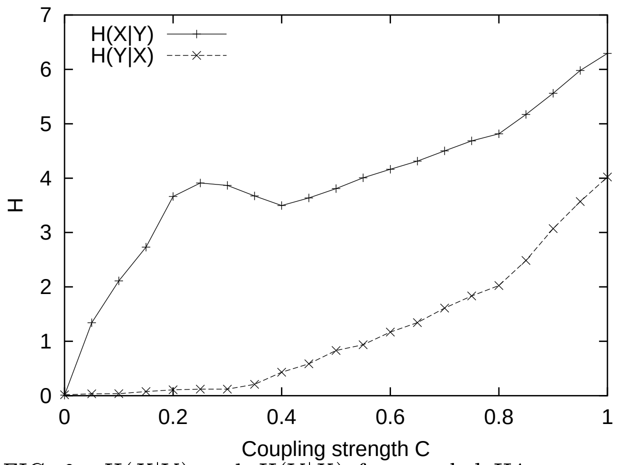

FIG. 2. $H(X|Y)$ and $H(Y|X)$ for coupled Hénon maps with $b_x = 0.3$ and $b_y = 0.1$.

$$S^{(k)}(X|Y) = \frac{1}{N}\sum_{i=1}^{N} \frac{R_i^{(k)}(X)}{R_i^{(k)}(X|Y)} \tag{5}$$

can be defined which takes values from nearly zero for independent systems to one for strongly dependent and identical systems. A measure with similar properties is

$$H^{(k)}(X|Y) = \frac{1}{N}\sum_{i=1}^{N} \log \frac{R_i^{(N-1)}(X)}{R_i^{(k)}(X|Y)} \tag{6}$$

which differs from S only in comparing the conditional distance to the mean distance to all other points and using the logarithm. Both quantities have proven to be quite useful in real data applications [3] and simple toy models [4]. While S and H show the same behavior in all cases studied here, H is more suitable to tell the "direction" of the coupling by its more pronounced asymmetry. We will show two simple examples in the following.

To illustrate the behavior of H we consider two unidirectionally coupled Hénon maps

$$\begin{aligned} {x_1}' &= 1.4 - x_1^2 + b_x x_2 \\ {x_2}' &= x_1 \end{aligned} \tag{7}$$

$$\begin{aligned} {y_1}' &= 1.4 - ((1-C)y_1 + Cx_1)y_1 + b_y y_2 \\ {y_2}' &= y_1 \end{aligned} \tag{8}$$

Note that the autonomous system X can use a different parameter b than the driven system Y. For the first example we use $b_x = b_y = 0.3$ and calculate H with an embedding dimension $m = 3$ and $k = 20$ nearest neighbors. Figure 1 shows $H(X|Y)$ and $H(Y|X)$ in dependence of the coupling strength C. For values $C \geq 0.7$ the two systems synchronize identically which can easily be seen in a sharp increase of H and the following equality of $H(X|Y)$ and $H(Y|X)$. This increase can also be seen with other dependence measures like mutual information (2) or linear correlation (1).

For weaker coupling we observe a strong asymmetry

$$H(X|Y) > H(Y|X)\,. \tag{9}$$

This behavior can also be seen in Figure 2 where for the driven system $b_y = 0.1$ is used and Eq.(9) holds for all $C > 0$. Quiroga et al. give further examples and find Eq.(9) holding generally [4].

The shown asymmetry of H is its main advantage over similar measures for dependence. The caveat of the above examples is that the two involved systems already show different dynamics on their own. They have e.g. different entropies, dimensions etc. and in all the above examples also $I(Y) \geq I(X)$ holds for all C. In this paper we want to address the question if the asymmetry of H arises from different properties of the individual time series or does indeed measure the direction of the coupling. To achieve this we have to construct systems with unidirectional coupling but otherwise completely identical properties.

Therefore, we consider a coupled map lattice $\{x_i^l\}, \quad l = 1, \ldots, L$ with periodic boundary conditions. In such a "ring" of L maps we have translation invariance and no lattice point is singled out. The unidirectional coupling is introduced as

$$x_{i+1}^l = (1-\varepsilon)f(x_i^l) + \varepsilon f(x_i^{l-1})\,. \tag{10}$$

Of course, we need $L > 2$ to get asymmetry and in the following examples we use $L = 100$ to ensure that causal influences in the backward direction are negligible. For $f(\cdot)$ the tent map $f(x) = 1 - |x - \frac{1}{2}|$ on the interval $[0,1]$ is used. Figure 3 shows that the asymmetry of H is also present for this system, while quantities like the entropy

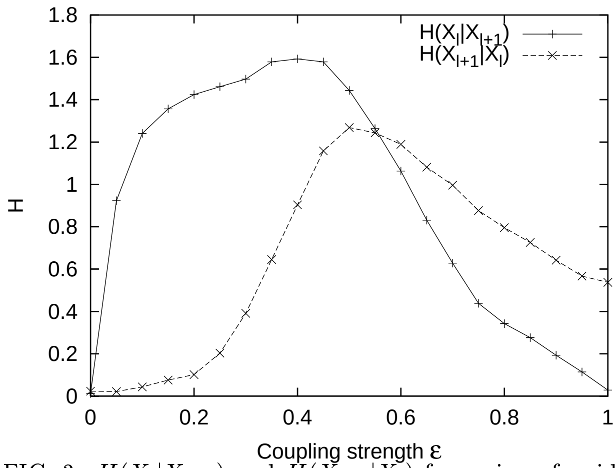

FIG. 3. $H(X_l|X_{l+1})$ and $H(X_{l+1}|X_l)$ for a ring of unidirectional coupled tent maps.

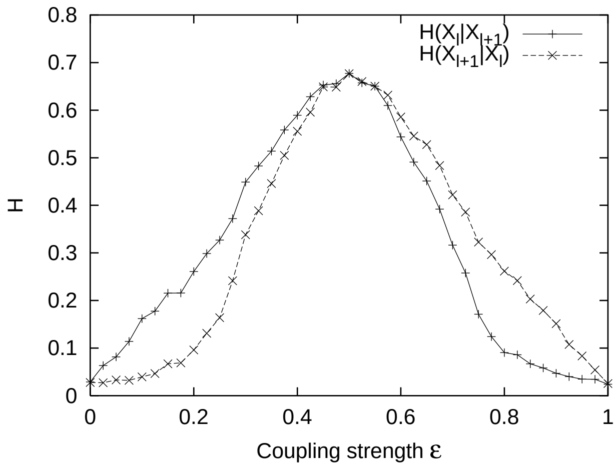

FIG. 4. As Figure 3, but without embedding.

I are the same for all lattice sites within numerical fluctuations. Thus the asymmetry H does not rely just on the individual properties of the two time series. Another interesting observation from Figure 3 is that Eq.(9) does not hold for all ε. There is a crossover at about $\varepsilon \approx 0.55$. And further, we see that $H(X_{l+1}|X_l)$ does not vanish for $\varepsilon = 1$ where the maps should be un-coupled and only be shifted by one lattice site in every iteration.

The reason for the later is that we used two dimensional embedding vectors ($m = 2$) to calculate H, i.e. we analyzed the dependence between the two vectors

$$\begin{pmatrix} x_i^{l-1} \\ x_{i+1}^{l-1} \end{pmatrix} \searrow \begin{pmatrix} x_i^{l} \\ x_{i+1}^{l} \end{pmatrix} \tag{11}$$

where the arrow symbolizes the map $f(\cdot)$ which for $\varepsilon = 1$ is the only direct influence between the components of the two vectors. If H is calculated without embedding ($m = 1$) we get Figure 4 and H vanishes for $\varepsilon = 1$ in both directions.

To understand the inversion of Eq.(9) for $\varepsilon > 0.5$ we observe that both x_{i+1}^{l-1} and x_{i+1}^{l} are influenced by x_i^{l-1} and one other neighboring lattice site. The coupling strength ε determines the corresponding weights. Neglecting the action of the map $f(\cdot)$ we can model this by a simple mixing in the following form

$$\begin{aligned} u_i &= \varepsilon\, y_i^{(1)} + (1-\varepsilon) y_i^{(2)} \\ v_i &= (1-\varepsilon) y_i^{(3)} + \varepsilon\, y_i^{(2)} \end{aligned} \tag{12}$$

where $Y^{(l)} = \{y_i^{(l)}\}$, $l = 1, 2, 3$, are iid. random numbers. In the following we shall show that even this toy model gives the same non–trivial results as the coupled map lattice. If we choose uniform distributions $p(Y^{(1)}) = p(Y^{(2)}) = p(Y^{(3)}) = 1$ for $Y^{(l)}$, the resulting $H(U|V)$ and $H(V|U)$ does in fact show almost the

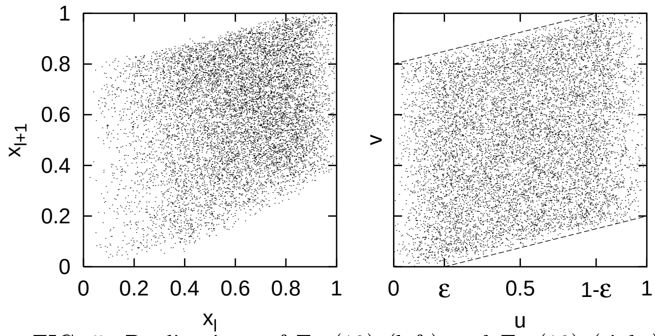

FIG. 5. Realizations of Eq.(10) (left) and Eq.(12) (right) for $\varepsilon = 0.2$.

same ε–dependence as the coupled map lattice shown in Figure 4. Comparing the distributions $p(X_l, X_{l+1})$ and $p(U, V)$ for fixed ε in Figure 5, we see that they are also quite similar.

Let us concentrate on the right figure to illustrate the behavior of the different distances contributing to $H(U|V)$ and $H(V|U)$. $R_i^{(N-1)}(U)$ and $R_i^{(k)}(U|V)$ are squared mean distances from (u_i, v_i) in the horizontal direction. While $R_i^{(N-1)}(U)$ averages over *all* points, $R_i^{(k)}(U|V)$ averages only over k points lying in a thin *horizontal* band around v_i containing its nearest neighbors on the v–axis. If v_i lies in the interval $[\varepsilon, 1-\varepsilon]$ these points are spread over the full u-interval $[0, 1]$. Hence, $R_i^{(k)}(U|V)$ and $R_i^{(N-1)}(U)$ are approximately of same order, independent from the fact that the corresponding distributions are not uniform. This changes if $v_i < \varepsilon$ or $v_i > 1-\varepsilon$ where the points in the band are bound to a smaller u-region and $R_i^{(k)}(U|V) < R_i^{(N-1)}(U)$. With this tendency to quotients greater than 1 in Eq.(6), the average over all points (u_i, v_i) yields a positive $H(U|V)$.

An analogous reasoning can be followed for $R_i^{(N-1)}(V)$ and $R_i^{(k)}(V|U)$. The squared mean distance from (u_i, v_i)

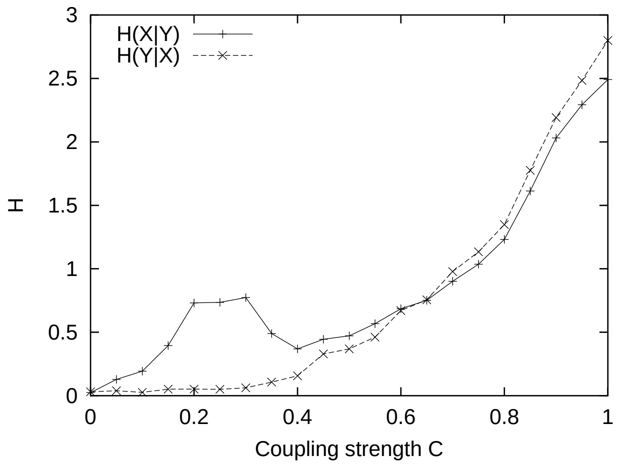

FIG. 6. As Figure 2, but without embedding.

to all other points $R_i^{(N-1)}(V)$ in vertical direction is in fact of the same order as $R_i^{(N-1)}(U)$. Both quantities only depend on their corresponding univariate probability distribution $p(U)$ resp. $p(V)$ and the symmetric form of Eq.(12) obviously implies $p(U) = p(V)$. $R_i^{(k)}(V|U)$ averages over k points lying in thin *vertical* band u_i. Again, $R_i^{(k)}(V|U)$ is of the same order as $R_i^{(N-1)}(V)$ if u_i lies in the interval $[\varepsilon, 1-\varepsilon]$ and deviates from it if u_i is outside this band. But, as can be seen from Figure 5, the deviation is much smaller. For small ε, $R_i^{(k)}(V|U)$ deviates from $R_i^{(N-1)}(V)$ by terms $\mathcal{O}(\varepsilon^2)$, while the analogous deviation of $R_i^{(k)}(U|V)$ from $R_i^{(N-1)}(U)$ is $\mathcal{O}(\varepsilon)$. Thus, $H(V|U) < H(U|V)$.

The above argumentation is only valid for small ε. The observed asymmetry of H around $\varepsilon = 0.5$ in Figure 4 can be best understood by Eq.(12). There, replacing ε by $1-\varepsilon$ is the same as exchanging U and V. This is, of course, only true if $p(Y^{(1)}) = p(Y^{(3)})$ resp. $p(X_{l-1}) = p(X_{l+1})$ which holds in both systems considered here.

Our examples show that H is indeed able to find dependencies and directional information from time series. But an interpretation of the two observed systems as "driver" and "response" would only be possible if the existence of a third external driving system can be excluded. In the example of simple mixing the asymmetry of H arises from different information contents contributed by a third system. For all considered unidirectionally coupled systems the asymmetry of H correctly detects the direction of the coupling if the coupling strength is small.

Further, we would like to notice that we considered scalar time series without embedding only for simplicity reasons. To see the necessity of embedding compare Figure 2 with Figure 6 where the scalar time series was used and the asymmetry of H is broken for $C > 0.6$. As for many other methods in non–linear time series analysis [5], a range of embedding dimensions should be applied and compared to get optimal results.

We like to thank Peter Grassberger, Thomas Schreiber, Jochen Arnhold and Rodrigo Quian Quiroga for useful discussions.

[1] D. Prichard and J. Theiler, *Generalized redundancies for time series analysis*, Physica D **84**, 476 (1995)

[2] N.F. Rulkov, M.M. Sushchik, L.S. Tsimring and H.D.I. Abarbanel, *Generalized synchronization of chaos in directionally coupled chaotic systems*, Phys. Rev. E **51**, 980 (1995)

[3] J. Arnhold, P. Grassberger, K. Lehnertz and C.E. Elger, *A Robust Method for Detecting Interdependencies: Application to Intracranially Recorded EEG*, Physica **D 134**, 419 (1999)

[4] R. Quian Quiroga, J. Arnhold and P. Grassberger, *Learning Driver-Response Relationships from Synchronization Patterns*, to appear in Phys. Rev. E (2000)

[5] H. Kantz and T. Schreiber, *Nonlinear time series analysis*, Cambridge University Press (1997).